15 Elementary Properties of Eigensystems

Eigenvalues and Eigenvectors For an $n \times n$ matrix A, scalars λ and vectors $x \neq 0$ ($x \in \mathbb{R}^n$) satisfying $A\boldsymbol{x} = \lambda\boldsymbol{x}$ are called ***eigenvalues*** and ***eigenvectors*** of A, respectively, and any such pair, $(\lambda, \boldsymbol{x})$, is called an ***eigenpair*** for A. The set of distinct eigenvalues, denoted by $\sigma(A)$, is called the ***spectrum*** of A.

- $\lambda \in \sigma(A) \Longleftrightarrow A - \lambda I$ is singular $\Longleftrightarrow$ $\Longleftrightarrow \det(A - \lambda I) = 0$.
- $\{x \neq 0 \,|\, x \in \ker(A - \lambda I)\}$ is the set of all eigenvectors associated with λ. From now on, $E(\lambda, A) := \ker(A - \lambda I)$ is called an ***eigenspace*** for A.

Characteristic Polynomial and Equation

- The ***characteristic polynomial*** of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is $p(\lambda) = \det(A - \lambda I)$. The degree of $p(\lambda)$ is n, and the leading term in $p(\lambda)$ is $(-1)^n\lambda^n$.
- The ***characteristic equation*** for A is $p(\lambda) = 0$.
- The eigenvalues of A are the solutions of the characteristic equation or, equivalently, the roots of the characteristic polynomial.
- Altogether, A has n eigenvalues, but some may be complex numbers (even if the entries of A are real numbers), and some eigenvalues may be repeated.
- If A contains only real numbers, then its complex eigenvalues must occur in conjugate pairs - i.e., if $\lambda \in \sigma(A)$, then $\overline{\lambda} \in \sigma(A)$.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote a given operator defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 3y \\ x + 5y \end{pmatrix}.$$

Find eigenvectors and eigenvalues of f.

2. Let (a) $A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.
Find characteristic polynomial of A, and corresponding eigenspaces.

3. It is given a matrix $A = \begin{pmatrix} 7 & -4 & 0 \\ a & -7 & b \\ 3 & -2 & 0 \end{pmatrix}$ which eigenvalues are -1 and 1. Find a parameters $a, b \in \mathbb{R}$ and find algebraic and geometrical multiplicities of all eigenvalues of A.

4. Find eigenvalues, eigenspaces and algebraic and geometrical multiplicities of all eigenvalues of A, if matrix A is given by

$$A = \begin{bmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

Multiplicities For $\lambda \in \sigma(A) = \{\lambda_1, \lambda_2, ..., \lambda_s\}$, we adopt the following definitions.

- The ***algebraic multiplicity*** of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $\operatorname{alg\,mult}_A(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1}...(x-\lambda_s)^{a_s} = 0$ is the characteristic equation for A.
- When $\operatorname{alg\,mult}_A(\lambda) = 1$, λ is called a ***simple eigenvalue***.
- The ***geometric multiplicity*** of λ is $\dim\ker(A-\lambda I)$. In other words, $\operatorname{geo\,mult}_A(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ.
- Eigenvalues such that $\operatorname{alg\,mult}_A(\lambda) = \operatorname{geo\,mult}_A(\lambda)$ are called ***semisimple eigenvalues*** of A.

5. Let

$$A = \begin{pmatrix} 1 & a & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & b & 0 & 1 \end{pmatrix}$$

be a given matrix. Find parameters a and b if it is known that A is singular matrix which all eigenvalues have algebraic multiplicity 2.

6. Let a denote some real number. Find all eigenvalues and corresponding eigenspaces for a given matrix of n-th order

$$\begin{bmatrix} 1+a & 1 & 1 & ... & 1 & 1 \\ 1 & 1+a & 1 & ... & 1 & 1 \\ 1 & 1 & 1+a & ... & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & ... & 1+a & 1 \\ 1 & 1 & 1 & ... & 1 & 1+a \end{bmatrix}.$$

7. Diagonalize given matrices

$$(a)\ A = \begin{pmatrix} 3 & 3 \\ 2 & 5 \end{pmatrix}; \quad (b)\ A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix};$$

(i.e. find invertible matrix P, with entries from $\mathbb{R}$, for which $P^{-1}AP = D$ hold, where D is diagonal matrix with entries from $\mathbb{R}$).

8. Find a real number λ such that $A = \begin{pmatrix} i & 1 \\ 2i & \lambda \end{pmatrix}$ has eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Is it possible to diagonalize A?

16 Generalized Eigenvectors and Nilpotnent Operators

Generalized Eigenvector Suppose $T \in \mathcal{L}(\mathcal{V})$ and λ is an eigenvalue of T. A vector $\boldsymbol{v} \in \mathcal{V}$ is called a ***generalized eigenvector*** of T corresponding to λ if $\boldsymbol{v} \neq 0$ and

$$(T - \lambda I)^j = 0$$

for some positive integer j.

Generalized Eigenspace, $G(\lambda, T)$ Suppose $T \in \mathcal{L}(\mathcal{V})$ and $\lambda \in \mathbb{F}$. The ***generalized eigenspace*** of T corresponding to λ, denoted $G(\lambda, T)$, is defined to be the set of all generalized eigenvectors of T corresponding to λ, along with the 0 vector.

Description of generalized eigenspaces Suppose $T \in \mathcal{L}(\mathcal{V})$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = \ker(T - \lambda I)^{\dim \mathcal{V}}$.

Linearly independent generalized eigenvectors Let $T \in \mathcal{L}(\mathcal{V})$. Suppose λ_1, λ_2, ..., λ_m are distinct eigenvalues of T and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_m$ are corresponding generalized eigenvectors. Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\}$ is linearly independent set.

1. Define $T \in \mathcal{L}(\mathbb{C}^3)$

$$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3).$$

(a) Find all eigenvalues of T, the corresponding eigenspaces, and the corresponding generalized eigenspaces.

(b) Show that $\mathbb{C}^3$ is the direct sum of generalized eigenspaces corresponding to the distinct eigenvalues of T.

2. Define $T \in \mathcal{L}(\mathbb{C}^2)$ by

$$T(w, z) = (z, 0).$$

Find all generalized eigenvectors of T.

3. Define $T \in \mathcal{L}(\mathbb{C}^2)$ by

$$T(w, z) = (-z, w).$$

Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.

Nilpotent An operator is called ***nilpotent*** if some power of it equals 0.

Nilpotent operator raised to dimension of domain is 0 Suppose $N \in \mathcal{L}(\mathcal{V})$ is nilpotent. Then $N^{\dim \mathcal{V}} = 0$.

Matrix of a nilpotent operator Suppose N is a nilpotent operator on $\mathcal{V}$. Then there is a basis of $\mathcal{V}$ with respect to which the matrix of N has the form

$$\begin{pmatrix} 0 & * & * & ... & * & * \\ 0 & 0 & * & ... & * & * \\ 0 & 0 & 0 & ... & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & * \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

here all entries on and below the diagonal are 0's.

4. Operator $T \in \mathcal{L}(\mathbb{R}^4)$ is defined on the following way

$$T(z_1, z_2, z_3, z_4) = (z_3, z_4, 0, 0)$$

(a) Check is T nilpotent operator.

(b) Find a basis $\mathcal{B}$ of space $\mathbb{R}^4$ such that

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(all entries on and below the diagonal are 0's).

5. Explain why the operator of differentiation on $\mathcal{P}_m(\mathbb{R})$ is nilpotent.

6. Suppose $T \in \mathcal{L}(\mathcal{V})$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

7. Suppose $T \in \mathcal{L}(\mathcal{V})$ and $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$. Prove that

$$G(\alpha, T) \cap G(\beta, T) = \{\mathbf{0}\}.$$

8. Suppose $T \in \mathcal{L}(\mathcal{V})$, m is a positive integer, and $\boldsymbol{v} \in \mathcal{V}$ is such that $T^{m-1}\boldsymbol{v} \neq 0$ but $T^m \boldsymbol{v} = 0$. Prove that

$$\boldsymbol{v}, T\boldsymbol{v}, T^2\boldsymbol{v}, ..., T^{m-1}\boldsymbol{v}$$

is linearly independent.

9. Suppose $N \in \mathcal{L}(\mathcal{V})$ is nilpotent. Prove that 0 is the only eigenvalue of N.

10. Prove or give a counterexample: The set of nilpotent operators on $\mathcal{V}$ is a subspace of $\mathcal{L}(\mathcal{V})$.